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(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FIFTH SEMESTER EXAMINATION, DECEMBER 2016
THIRD YEAR [BATCH 2014-17]
MATHEMATICS [Honours]
Paper : V

Date : 14/12/2016
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book for each Group]

Group – A

(*Answer any five questions*) [5×10]

1. a) Define a normal subgroup of a group. Give example of a subgroup which is not normal.
   b) Let H be a proper subgroup of a group G and $a \in G,\ a \notin H$. Suppose that for all $b \in G$, either $b \in H$ or $Ha = Hb$. Show that H is a normal subgroup of G.
   c) Prove that there are only two groups (upto isomorphism) of order 6. [3+2+5]

2. a) Let $G = \{a \in \mathbb{R} : -1 < a < 1\}$, show that the group $(G,*)$ and the group $(\mathbb{R},+)$ are isomorphic where $a * b = \dfrac{a+b}{1+ab}$ for all $a,b \in G$.
   b) Show that additive group $(Z, +)$ cannot be expressed as internal direct product of two of its subgroups.
   c) Determine the class equation of a non-abelian group of order six. [5+2+3]

3. a) Let G be a cyclic group of order mn where m, n are positive intergers with $\gcd(m,n)=1$. Show that $G \cong \mathbb{Z}_m \times \mathbb{Z}_n$.
   b) Prove that any group of order p^2 is abelian where p is prime.
   c) How many elements of order 7 are there in a group of order 28? Justify your answer. [5+3+2]

4. a) State and prove Cauchy's theorem on finite group.
   b) Prove that a group of order 90 is not simple. [5+5]

5. a) Let D and D' be two isomorphic integral domains through an isomorphism f. Show that f can be uniquely extended to an isomorphism of F onto F', where F and F' are quotient fields of D and D' respectively.
   b) Let R be a commutative ring with identity $1 \neq 0$. Then prove that a proper ideal P of R is prime if and only if R/P is an integral domain. [5+5]

6. a) Let R be a P.I.D then prove that every $a \in R$ which is not invertible can be expressed as a product of irreducible elements.
   b) In the ring $R = \left\{a + b\sqrt{-5} \,/\, a,b \in Z\right\}$, show that $1+2\sqrt{-5}$ is irreducible but not prime. [5+5]

7. a) Define a polynomial ring.
   Prove that if R is an integral domain, then $R[x]$ is also an integral domain.
   b) Show that any finite integral domain is a field.
   c) Let R be a ring such that $a^2 = a,\ \forall a \in R$, prove that the characteristic of R is two. [5+3+2]

8. a) Let R be a commutative ring with identity and $a,b \in R$. Show that $I = \{ra + tb : r,t \in R\}$ is an ideal of R.

b) Suppose F is a field and there is a ring homomorphism from $\mathbb{Z}$ onto F. Show that $F \simeq \mathbb{Z}_p$ for some prime number p.

c) Let K be a field and $p(x) \in K[x]$. Let P be the principal ideal generated by $p(x)$. Prove that $\frac{K[x]}{P}$ is an integral domain if and only if $\frac{K[x]}{P}$ is a field. [3+2+5]

<u>Group – B</u>

<u>Unit - I</u>

(*Answer <u>any six</u> questions*) [6×5]

9. a) Consider the function

$$f(x,y) = x\sin\frac{1}{y} + y\sin\frac{1}{x}, \quad xy \neq 0$$
$$= \quad 0, \quad xy = 0$$

Prove that the repeated limits do not exist but the double limit exists.

b) Check the existence of the double limit

$$\lim_{(x,y)\to(0,0)} \left(\frac{\sin x + \sin 2y}{\tan 2x + \tan y} \right)$$ [3+2]

10. Prove that a sufficient condition that a function f be continuous at (a, b) is that one of the partial derivatives exist and is bounded in a neighbourhood of (a, b) and the other exists at (a, b). [5]

11. Show that for the following function, the sufficient conditions for the differentiability at a point do not hold but the function is differentiable at that point: [5]

$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y}, & xy \neq 0 \\ x^2 \sin\frac{1}{x}, & x \neq 0 \\ y^2 \sin\frac{1}{y}, & y \neq 0 \\ 0, & x = 0 = y \end{cases}$$

12. State and prove Schwarz's Theorem on the commutativity of order of partial derivatives. [5]

13. Transform the equation $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = z^2$, taking $u = x,\ v = \frac{1}{y} - \frac{1}{x}$ for the new independent variables and $w = \frac{1}{z} - \frac{1}{x}$ for the new function. [5]

14. Given $f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$ where f is differentiable function and $f(0) = 0$ and $f(x)f(y) \neq 1$, show that $f(t) = \tan \alpha t,\ \alpha$ is constant. [5]

15. a) Examine the applicability of implicit function theorem for $f(x,y) = 0$, where $f(x,y) = xy\sin x + \cos y$ in the neighbourhood of $(0, \frac{\pi}{2})$.

b) Show that

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2 + y^4}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is not differentiable at (0,0). [3+2]

16. Let (x, y) approach $(0,0)$ along $y = -x$. Using Taylor's theorem find limit of $\dfrac{\sin xy + xe^x - y}{x\cos y + \sin 2y}$. [5]

17. Show that the function

$$u = \phi(xy) + \sqrt{xy}\,\psi\left(\tfrac{y}{x}\right)$$

satisfies the equation $x^2 \dfrac{\partial^2 u}{\partial x^2} - y^2 \dfrac{\partial^2 u}{\partial y^2} = 0$ where u is twice differentiable function of x & y. [5]

Unit - II

(*Answer any four questions*) [4×5]

18. a) State Second Mean-Value Theorem of Integral Calculus due to Bonnet. Using it show that there exists a point $\alpha \in [0,\pi]$ such that $\int_0^{\pi} e^{-x}\cos x\, dx = \sin\alpha$.

b) Show that $\dfrac{\pi^3}{24\sqrt{2}} < \int_0^{\pi/2} \dfrac{x^2}{\sin x + \cos x}dx < \dfrac{\pi^3}{24}$. [3+2]

19. The graph of a function $f:[a,b] \to R$ is rectifiable iff it is of bounded variation on $[a, b]$. Show that the graph of f given by

$f(x) = x\sin\left(\frac{\pi}{x}\right),\ x \neq 0$ & $f(0) = 0$ is not rectifiable. [3+2]

20. a) If f be continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, for $x \in [a,b]$, then show that

$$F'(x) = f(x) \text{ for all } x \in [a,b].$$

b) Give an example of a function f which is Riemann integrable without having a primitive. [3+2]

21. a) Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a,b]$. If $\exists$ a point c in $]a,b[$ such that f is continuous at c and $f(c) > 0$ then prove that $\int_a^b f(x)dx > 0$.

b) If f is a continuous function on $[a, b]$ such that $\int_a^b f^2(x)dx = 0$ then prove that $f(x) = 0$ for all $x \in [a,b]$. [3+2]

22. a) Let $f_n:[a,b] \to \mathbb{R}$ be Riemann integrable for each $n \in \mathbb{N}$. If the sequence $\{f_n\}$ converges uniformly to a function f on $[a, b]$ then show that f is also Riemann integrable on $[a, b]$.

b) Show that $-\dfrac{1}{2} < \int_0^1 \dfrac{x^3\cos 5x}{2+x^2}dx < \dfrac{1}{2}$. [3+2]

23. a) Let f be bounded and integrable on $[a, b]$. If there exists a function ϕ such that $\phi'(x) = f(x)$ for every $x \in [a,b]$, then prove that $\int_a^b f(x)dx = \phi(b) - \phi(a)$.

b) Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\int_{x^2}^{x^4} \sin\sqrt{t}\,dt}{x^3}\right)$. [3+2]

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